DISCRETE-CONTINUUM TRANSITIONS. II. GENERAL CASE

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Abstract

A general formula for bound-continuous transition form factors is derived. It is shown that these form factors can be represented in the form of finite sum of terms with simple analytical structure.

In the previous paper [1] it have been shown that the form factors of transitions from nS $(n00)$ - states of hydrogenlike atoms [2] to the state of continuous spectra with definite value of relative momenta $\vec{p}$ may be expressed in the terms of the classical polynomials in a rather simple way. Below this result is generalized for the case of transition from arbitrary initial bound states.

The transition form factors are defined as follows:

$$S_{fi}(\vec{q}) = \int \psi_f^*(\vec{r}) e^{i\vec{q}\vec{r}} \psi_i(\vec{r}) d^3r \,, \tag{1}$$

Here, $\psi_{i(f)}$ are the wave functions of initial (final) states.

According to [3] (see also [4]), the final state wave function must be choose in the form

$$\psi_i(\vec{r}) \equiv \psi_{n00}(\vec{r}) \,. \tag{2}$$

For arbitrary initial bound state

$$\psi_i(\vec{r}) = \psi_{nlm}(\vec{r}) = Y_{lm}\left(\frac{\vec{r}}{r}\right) R_{nl}(r) \,, \tag{3}$$

$$\begin{aligned} R_{nl}(r) &= \frac{2\omega^{\frac{3}{2}}}{\Gamma(2l+2)} \left[\frac{\Gamma(n+l+1)}{n\Gamma(n-l)}\right]^{\frac{1}{2}} \cdot (2\omega r)^l \cdot \Phi(-n+l+1, 2l+2; 2\omega r) \cdot \exp(-\omega r) \\ &= 2\omega^{\frac{3}{2}} \left[\frac{\Gamma(n-l)}{n\Gamma(n+l+1)}\right]^{\frac{1}{2}} \cdot (2\omega r)^l \cdot L_{n-l-1}^{2l+1}(2\omega r) \cdot \exp(-\omega r) \,, \\ &\qquad \omega = \mu\alpha/n \,, \end{aligned}$$

where L_{n-l-1}^{2l+1} are the associated Laguerre polynomials.

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Making use of the recurrence relations [5]

$$L_k^{\lambda+1}(x) = \frac{1}{x}\left[(k+\lambda+1)L_{k-1}^{\lambda}(x) - (k+1)L_k^{\lambda}(x)\right] \tag{4}$$

and the representation of the Laguerre polynomials in terms of the generating function

$$L_k^{\lambda}(x) = \Delta_z^{(k)}\left[(1-z)^{-(\lambda+1)}\exp\left(\frac{xz}{z-1}\right)\right], \tag{5}$$

where operator $\Delta_z^{(k)}$ is defined as follows

$$\Delta_z^{(k)}\left[f(z)\right] = \frac{1}{k!}\left(\frac{d^k}{dz^k}f(z)\right)\bigg|_{z=0}, \tag{6}$$

let us rewrite the radial part of initial state wave function in the form

$$R_{nl} = \frac{\omega^{\frac{1}{2}}}{r}\left[\frac{\Gamma(n-l)}{n\Gamma(n+l+1)}\right]^{\frac{1}{2}} \cdot (2\omega r)^l \cdot \left[(n+l)\Delta_z^{(n-l-2)} - (n-l)\Delta_z^{(n-l-1)}\right] \tag{7}$$

$$\times\left[(1-z)^{-(l+1)}\exp\left(-\omega(z)r\right)\right],$$

$$\omega(z) = \omega\cdot(1+z)(1-z)^{-1} \tag{8}$$

more convenient for the further calculations.

Then it is not difficult to see that transition form factors (1) may be represent as a linear combination of the quantities

$$I_{lm}^{j} = \Delta_z^{(j)}\left[(1-z)^{-(2l+1)}J_{lm}(\vec{q},\vec{p},z)\right], \tag{9}$$

$$J_{lm}(\vec{q},\vec{p},z) = \int\frac{d^3r}{r}Y_{lm}\left(\frac{\vec{r}}{r}\right)\Phi\left[i\xi,1;i(pr+\vec{p}\vec{r})\right] \tag{10}$$

$$\times\exp[i(\vec{q}-\vec{p})\vec{r}-\omega(z)\cdot r]\cdot(2\omega r)^l\cdot\exp\left[-\omega(z)r\right],$$

$$j = n-l-2,\ n-l-1\,.$$

In order to calculate (10), it is useful to represent the hypergeometrical function in (3) in the form

$$\Phi\left[i\xi,1;i(pr+\vec{p}\vec{r})\right] = -\frac{1}{2\pi i}\oint_C(-t)^{i\xi-1}(1-t)^{-i\xi}\cdot\exp[i\cdot t(pr+\vec{p}\vec{r})]dt\,. \tag{11}$$

Using the following relations

$$\exp(i\vec{\tau}\vec{r}) = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}i^lY_{lm}\left(\frac{\vec{\tau}}{\tau}\right)Y_{lm}^{*}\left(\frac{\vec{r}}{r}\right)j_l(\tau r), \tag{12}$$

$$j_l(x) = \sqrt{\frac{\pi}{2x}}J_{l+\frac{1}{2}}(x), \tag{13}$$

$$\int_0^\infty r^{l+\frac{1}{2}} J_{l+\frac{1}{2}}(\tau r) e^{-\bar{\omega}\cdot r} dr = \frac{(2\tau)^{l+\frac{1}{2}}\Gamma(l+1)}{\sqrt{\pi}(\tau^2+\bar{\omega}^2)^{l+1}}\,, \tag{14}$$

where

$$\vec{\tau} = \vec{q} - \vec{p}(1-t), \quad \bar{\omega} = \omega(z) - ip\cdot t\,, \tag{15}$$

after simple calculations we find

$$J_{lm}(\vec{q},\vec{p},z) \;=\; -\frac{\Gamma(l+1)}{2\pi i}\oint_C dt(-t)^{i\xi-1}(1-t)^{-i\xi}\cdot\frac{4\pi(4i\omega)^l Y_{lm}\left(\vec{\tau}/\tau\right)\tau^l}{(\tau^2+\bar{\omega}^2)^{l+1}}\,. \tag{16}$$

It is easy to check that

$$\tau^2+\bar{\omega}^2 = a(1-t)+c\cdot t\,,$$
$$a = \omega^2(z)+\vec{\Delta}^2\,,\quad c = [\omega(z)-ip]^2+q^2\,. \tag{17}$$

Further, according to [6], we get

$$Y_{lm}\left(\frac{\vec{\tau}}{\tau}\right)\tau^l = \sum_{l_1=0}^{l} q^{l_1}(-p)^{l-l_1} \tag{18}$$

$$\times\left[\frac{4\pi\Gamma(2l+2)}{\Gamma(2l_1+2)\Gamma(2l-2l_1+2)}\right]^{\frac{1}{2}}\cdot(1-t)^{l-l_1}\left[Y_{l_1}\left(\frac{\vec{q}}{q}\right)\otimes Y_{l-l_1}\left(\frac{\vec{p}}{p}\right)\right]_{lm}\,,$$

$$\left[Y_{l_1}\left(\frac{\vec{q}}{q}\right)\otimes Y_{l-l_1}\left(\frac{\vec{p}}{p}\right)\right]_{lm} \;=\; \sum_{m_1+m_2=m} C^{lm}_{l_1m_1(l-l_1)m_2}\cdot Y_{l_1m_1}\left(\frac{\vec{q}}{q}\right)\cdot Y_{(l-l_1)m_2}\left(\frac{\vec{p}}{p}\right)\,. \tag{19}$$

Taking into account (17) and (18), it is easy to see that (16) is the superposition of the quantities

$$-\frac{1}{2\pi i}\oint_C \frac{t^{i\xi-1}(1-t)^{-i\xi+l-l_1}}{[a(1-t)+ct]^{l+1}} \tag{20}$$

$$\begin{aligned}
&= a^{-(l+1)}\frac{\Gamma(1-i\xi+l-l_1)}{\Gamma(1-i\xi)}F\left(i\xi,l+1;l-l_1+1;1-c/a\right)\\
&= a^{i\xi-l-1}c^{-i\xi}\frac{\Gamma(1-i\xi+l-l_1)}{\Gamma(1-i\xi)}F\left(i\xi,-l_1;l-l_1+1;1-a/c\right)\\
&= a^{i\xi-l-1}c^{-i\xi}\frac{\Gamma(l-l_1+1)\Gamma(l+1-i\xi)}{\Gamma(l+1)\Gamma(1-i\xi)}F(i\xi,-l_1;i\xi-l;a/c)\\
&= \sum_{s=0}^{l_1}(-1)^{l-s}\frac{\Gamma(i\xi+s)\Gamma(l_1+1)}{\Gamma(l_1-s+1)\Gamma(i\xi-l+s)\Gamma(s+1)\Gamma(l+1)}a^{i\xi+s-l-1}c^{-s-i\xi}\\
&= (1-z)^{2l+2}\sum_{s=0}^{l_1}(-1)^{l-s}\frac{\Gamma(i\xi+s)\Gamma(l_1+1)}{\Gamma(l_1-s+1)\Gamma(i\xi-l+s)\Gamma(s+1)\Gamma(l+1)}\\
&\qquad\times D_1^{i\xi+s-l-1}D_2^{-s-i\xi}\,,
\end{aligned}$$

where $D_{1,2}$ are defined as follows:

$$D_1 = (1+z^2)(\omega^2+\vec{\Delta}^2)-2z(\vec{\Delta}^2-\omega^2)\,, \tag{21}$$

$$D_2 = (\omega - ip)^2 + q^2 - 2z(q^2 - p^2 - \omega^2) + z^2[(\omega + ip)^2 + q^2]\,.$$

The further calculations are the same as in [1].

Omitting the simple but cumbersome algebra, let us present the final expression for transition form factors:

$$S_{\vec{p},nlm}(\vec{q}) \;=\; 4\pi \cdot 2^{2l} i^l \omega^{l+\frac{1}{2}} \left[\frac{\Gamma(n-l)}{n\Gamma(n+l+1)}\right]^{\frac{1}{2}} \tag{22}$$

$$\times \sum_{s=0}^{l} G_{lms}(\vec{p},\vec{q}) H_{nls}(\vec{p},\vec{q}) \cdot (\omega^2 + \Delta^2)^{i\xi+s-l-1}[(\omega - ip)^2 + q^2]^{-s-i\xi}\,;$$

$$G_{lms}(\vec{p},\vec{q}) \;=\; (-1)^{l-s} \frac{\Gamma(i\xi + s)}{\Gamma(i\xi - l + s)\Gamma(s+1)} \tag{23}$$

$$\times \sum_{l_1=s}^{l} \left[\frac{4\pi\Gamma(2l+2)}{\Gamma(2l_1+2)\Gamma(2l-2l_1+2)}\right]^{\frac{1}{2}} \cdot \frac{\Gamma(l_1+1)}{\Gamma(l_1-s+1)} q^{l_1}(-p)^{l-l_1}$$

$$\times \left[Y_{l_1}\left(\frac{\vec{q}}{q}\right) \otimes Y_{l-l_1}\left(\frac{\vec{p}}{p}\right)\right]_{lm}\,;$$

$$H_{nls}(\vec{p},\vec{q}) \;=\; (n+l)F_{n_1ls}(\vec{p},\vec{q}) - (n-l)F_{n_2ls}(\vec{p},\vec{q})\,; \tag{24}$$

$$n_1 = n - l - 1\,, \quad n_2 = n - l - 2\,;$$

$$F_{n_{1(2)}ls}(\vec{p},\vec{q}) \;=\; \frac{\Gamma(l - s + \frac{1}{2} - i\xi)}{\Gamma(2l - 2s + 1 - 2i\xi)} \sum_{k=0}^{n_{1(2)}} w^k C_k^{(i\xi+s)}(v) \tag{25}$$

$$\times \frac{\Gamma(n_{1(2)} - k + 2l - 2s + 1 - 2i\xi)}{\Gamma(n_{1(2)} - k + l - s + \frac{1}{2} - i\xi)} \cdot P_{n_{1(2)}-k}^{(l-s-\frac{1}{2}-i\xi,\,l-s+\frac{1}{2}-i\xi)}(u)\,.$$

Thus, the form factors for transition from arbitrary bound states of hydrogenlike atoms to the "$\vec{p} - state$" of continuous spectra are represented as the superposition of finite number of terms with simple analytical structure and can be evaluated numerically with arbitrary degree of accuracy.

Eqs. (22)-(25) are the generalization of the results of [7, 1].

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